THE UNIVERSITY OF NORTH CAROLINA
CHAPEL HILL

COMPUTATION CENTER

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July 16, 1965

National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama 35812

Attention: Mr. Harold Graham, PR-EC

Gentlemen:

Enclosed is a detailed progress report of the work performed at the UNC Computation Center in connection with NASA Contract NAS8-1528 in the period 1 October 1964 to 21 April 1965.

Sincerely yours,

James W. Hanson

James W. Hanson
Director

COMPUTATION CENTER
UNIVERSITY OF NORTH CAROLINA
CHAPEL HILL, NORTH CAROLINA

PROGRESS REPORT

1 OCTOBER 1964 -- 21 APRIL 1965

JAMES W. HANSON

CONTRACT NAS8-1528

PREPARED FOR GEORGE C. MARSHALL SPACE FLIGHT CENTER, NASA
HUNTSVILLE, ALABAMA

When Contract NAS8-1528 expired on 30 September 1964 the UNC Computation Center was involved in three computer-oriented studies and two studies in the general field of differential equations. Specifically, these studies were:

I. Extension and application of previously developed algorithms for obtaining multi-variable approximations by linear programming techniques.

II. Development of algorithms for performing symbolic mathematics by computer.

III. Development of methods of determining best prediction equations by linear programming.

IV. Control space studies.

V. Development of a linear algebraic formulation of the time optimal and two-point boundary condition problems.

At this time the Computation Center was encouraged to continue its efforts in all of the above areas with the understanding that the above contract was to be extended. On October 21 and 22, 1965 Center personnel attended on invitation, the Twentieth Technical Meeting at MSFC between NASA-Marshall Space Flight Center and Contractor Representatives concerning "Guidance and Space Flight Theory" contracts. An informal report was presented at the Large Computer Exploitation Group Meeting concerning the work being performed by Dr. Clay C. Ross, Jr. Verbal progress reports were also made to designated contacts on the NASA staff.

In December, 1964 a conference was held at NASA-Marshall Space Flight Center concerning the future of this work. It was decided at that time that the Computation Center would no longer work in the areas I, III, IV, V listed above. The contract proposal was modified so as to cover only digital computer applications to symbol manipulation problems and it was understood that a new contract would be issued as soon as possible. At this point all work was terminated except that involving problems of symbol manipulation.

Enclosure I summarizes the work completed in Control Space studies at the time this work was terminated in December, 1964.

Enclosure II summarizes the work completed on the development of a linear algebraic formulation of the time optimal and two-point boundary condition problems at termination in December, 1964.

Enclosure III summarizes the work completed on the development of methods for determining best prediction equations by linear programming at termination in December, 1964.

On February 3 and 4, 1965 Center personnel attended, on invitation, the Twenty-first Technical Meeting at NASA-Marshall Space Flight Center. Verbal progress reports on the studies being conducted in mathematical symbol manipulation were made at this meeting.

Enclosures IV and V summarize the work completed on the problems of mathematical symbol manipulation by large-scale digital computers from 1 October 1964 up to the issuance of the new contract, NAS8-20106 on 22 April 1965.

Saturn 931-project #10000

During the period 1 October 1964 and 21 April 1965 the following man-hours were expended on the above efforts:

October, 1964	330.5	man-hours
November, 1964	314.4	" "
December, 1964	244.1	" "
January, 1965	137.9	" "
February, 1965	137.9	" "
March, 1965	158.6	" "
April 1-21, 1965	82.1	" "

CONTROL SPACE

I. Summary.

The control space is the set of all trajectories obtainable using the available controls. In this study, it is assumed that in addition to the usual thrust angle control, there is another control, D, a continuous function which permits the thrust to assume values intermediate to the on and off positions. It is shown that the addition of D to the present thrust-angle control makes the associated control space convex. Inferences are drawn for the conventional case that $D \equiv 1$ (no reduction in thrust); and physical implications of the results are discussed. The flat earth problem is assumed.

II. Introduction.

In the conventional flat earth problem it is difficult to characterize the control space, because the available control, disregarding choice of cutoff time enters the problem in such a nonlinear way as to prevent the associated space from being convex.

In this paper it is assumed that in addition to φ, the control which determines the angle of rocket thrust, there is a "thrust baffle," D, which permits the amount of thrust to be varied continuously between its off position and its maximum value. D multiplies the thrust, T, so "off" corresponds to $D = 0$, whereas "on" in the conventional problem, or "maximum thrust" corresponds to $D = 1$. It is assumed that D does not affect the rate of fuel consumption, although the results presented are valid if it is, alternatively, assumed that the rate of fuel consumption is proportional to the control D.

It is found that the space of trajectories obtained from control pairs of the form (φ, D) is convex.

Convexity in the control space is desirable for several reasons. Certain problems in guidance are simplified, for instance. But more important, in the present context, is that convexity of the control space permits us to give an almost complete characterization of the control space by computing a finite number of trajectories which lie on the boundary. Then at each time, t , the permissible vehicle coordinates are contained in a polygon in E_4 determined by the computed trajectories.

If the control space is convex, then the subspace of all trajectories which successfully complete the mission with cutoff time, τ , is also convex; and the density of such trajectories at given points in the vehicle coordinate space can be easily computed. Such computations would form the basis of a reliability study. It is obviously desirable to choose a trajectory such that the given mission can be successfully completed from all the vehicle coordinate points in a fairly large neighborhood of that trajectory, and without great loss of time. Since the time-optimal trajectory is unique, it is on the boundary of the control space; and it is by no means clear, a priori, that a given mission can be completed at all from points near the time optimal trajectory, let alone within a feasible period of time. It may well be that with only a slight sacrifice in time optimality, a great increase in reliability can be obtained. One of the purposes of this research was to facilitate such a study.

III. Notation.

1. If D is a continuous function of time, with range the unit interval, and φ is a continuous function of time which describes the angle of thrust made with the y-axis, then the trajectory $W = \{x,y,u,v\}$ associated with the controls $\{\varphi,D\}$ satisfies the following differential equation:

1, a. $\dot{x}(t) = u(t)$

b. $\dot{y}(t) = v(t)$

c. $\dot{u}(t) = D(t)\ T(t) \sin \varphi(t)$

d. $\dot{v}(t) = D(t)\ T(t) \cos \varphi(t) - g$

2. We denote by C_0 all the trajectories satisfying equations 1,a,b,c,d. together with the initial conditions, $x(0) = y(0) = u(0) = v(0) = 0$, where $\{\varphi,D\}$ is an allowable control.

3. We denote by C_τ all the trajectories satisfying equations III. 1,a,b,c,d. together with end conditions $W(\tau) = P(\tau)$, where P

is the parabola in vehicle coordinate space on which the vehicle coordinate space on which the vehicle completes its mission by unpowered flight.[1] It should be noted that the <u>initial</u> conditions are not specified for trajectories in C_τ, for any $\tau > 0$; but rather the condition is made that cutoff at time τ will cause the vehicle on any trajectory in C_τ to reach the desired destination.

4. We denote by $C_0(t)$ the set of points X in E_4 (the vehicle coordinate space) with the property that there is a trajectory W in C_0 such that $W(t) = X$. If $\tau > 0$, and $0 \leq t \leq \tau$, then $C_\tau(t)$ is the set of points X in E_4 with the property that there is a trajectory, W, in C_τ such that $W(t) = X$.

5. We distinguish different trajectories by use of subscripts, and the associated components and contols will be given the same subscripts. Thus, if p is a number, then $W_p = \{x_p, y_p, u_p, v_p\}$ is the tajectory associated with the control pair (φ_p, D_p).

6. The set of trajectories which satisfy the initial conditions (III,2) and successfully complete the mission, with cutoff time τ, is $C_0 \cap C_\tau$.

If $\mathcal{J}$ is the set of allowable cutoff times, then the set of all trajectories with which the mission is successfully completed is

$$C_0 \cap (\cup \{C_\tau \mid \tau \text{ in } \mathcal{J} \}) .$$

7. $\tilde{C}_0$ is the subspace of C_0 generated by controls of the form $\{\varphi, 1\}$.

$\tilde{C}_\tau$ is the subspace of C_τ generated by controls of the form $\{\varphi, 1\}$.

IV. <u>Convexity</u>.

Suppose each of W_1 and W_2 is a trajectory in C_0 (or C_τ) and σ is in $(0,1)$. We wish to compute a control pair (φ_3, D_3) whose associated trajectory W_3 is such that

$$W_3 = \sigma W_1 + (1 - \sigma) W_2 \qquad (1)$$

It is clear that the initial (or cutoff) conditions are satisfied.

The defining equation for D_3 will be

$$D_3(t) = [\ (\sigma D_1(t))^2 + ((1-\sigma)D_2(t))^2 + 2\sigma\ (1-\sigma)\ D_1(t)D_2(t)\ \cos\ (\varphi_1(t)-\varphi_2(t))\ ]^{1/2} \quad . \tag{2}$$

It is clear from inspection that the function D_3 so defined is well-defined, and continuous if D_1, D_2, φ_1, and φ_2 are continuous.

The function φ_3 is defined by the pair of equations:

$$\sin\ (\varphi_3(t)) = \sigma \left(\frac{D_1(t)}{D_3(t)}\right) \sin\ (\varphi_1(t)) + (1-\sigma) \left(\frac{D_2(t)}{D_3(t)}\right) \sin\ (\varphi_2(t)) \tag{3a}$$

$$\cos\ (\varphi_3(t)) = \sigma \left(\frac{D_1(t)}{D_3(t)}\right) \cos\ (\varphi_1(t)) + (1-\sigma) \left(\frac{D_2(t)}{D_3(t)}\right) \cos\ (\varphi_2(t)) \tag{3b}$$

It remains to be demonstrated that equations IV.3.a,b. define a function φ_3, and that the trajectory W_3 associated with the control (φ_3, D_3) does, in fact, satisfy equation IV.1.

It is clear that φ_3 is defined by equations IV.3.a,b. if and only if the sum of the squares of the functions on the right are equal to one, for each time t. Thus

$$\begin{aligned}
&\left[\sigma \left(\frac{D_1}{D_3}\right) \sin \varphi_1 + (1-\sigma) \left(\frac{D_2}{D_3}\right) \sin \varphi_2\right]^2 \\
&\quad + \left[\sigma \left(\frac{D_1}{D_3}\right) \cos \varphi_1 + (1-\sigma) \left(\frac{D_2}{D_3}\right) \cos \varphi_2\right]^2 \\
&\quad = \sigma^2 \left(\frac{D_1}{D_3}\right)^2 \sin^2 \varphi_1 + (1-\sigma)^2 \left(\frac{D_2}{D_3}\right)^2 \sin^2 \varphi_2 \\
&\qquad \left(\frac{D_1}{D_3}\right)^{2} \cos^2 \varphi_1 + (1-\sigma)^2 \left(\frac{D_2}{D_3}\right)^2 \cos^2 \varphi_2 \\
&\quad + 2\sigma\ (1-\sigma) \left(\frac{D_1 D_2}{D_3^2}\right) (\sin \varphi_1 \sin \varphi_2 + \cos \varphi_1 \cos \varphi_2) \\
&\quad = \left(\frac{1}{D_3}\right)^2 \left[(\sigma D_1)^2 + ((1-\sigma)D_2)^2 + 2\sigma\ (1-\sigma)D_1 D_2 \cos\ (\varphi_1 - \varphi_2)\right] \\
&\quad = 1 \quad , \quad \text{for each time } t.
\end{aligned} \tag{4}$$

Now when the defining equations for $\alpha W_1 + (1-\alpha)W_2$ are written, we see that:

$$\frac{d}{dt}(\alpha u_1 + (1-\alpha)u_2) \tag{5a}$$

$$= \alpha\dot{u}_1 + (1-\alpha)\dot{u}_2$$

$$= \alpha D_1 T \sin \varphi_1 + (1-\alpha)D_2 T \sin \varphi_2$$

$$= D_3 T \sin \varphi_3 = \dot{u}_3 \quad ,$$

and since

$$\alpha u_1(0) + (1-\alpha)u_2(0) = u_3(0) = 0$$

(or $\alpha u_1(\tau) + (1-\alpha)u_2(\tau) = u_1(\tau) = u_2(\tau) = u_3(\tau)$) , we have:

$$u_3 \equiv \alpha u_1 + (1-\alpha)u_2$$

$$\alpha\dot{v}_1 + (1-\alpha)\dot{v}_2 \tag{5b}$$

$$= \alpha D_1 T \cos \varphi_1 + (1-\alpha)D_2 T \cos \varphi_2$$

$$= D_3 T \cos \varphi_3 = \dot{v}_3 \quad ,$$

and since

$$\alpha v_1(0) + (1-\alpha)v_2(0) = v_3(0) = 0$$

(or $\alpha v_1(\tau) + (1-\alpha)v_2(\tau) = v_1(\tau) = v_2(\tau) = v_3(\tau)$) , we have

$$v_3 = \alpha v_1 + (1-\alpha)v_2$$

$$\alpha\dot{x}_1 + (1-\alpha)\dot{x}_2 = \alpha u_1 + (1-\alpha)u_2 = u_3 = \dot{x}_3 \tag{5c}$$

$$\alpha\dot{y}_1 + (1-\alpha)\dot{y}_2 = \alpha v_1 + (1-\alpha)v_2 = v_3 = \dot{y}_3 \tag{5d}$$

and it follows that

$$\alpha W_1 + (1-\alpha)W_2 \equiv W_3$$

since both sides of equation IV.6. satisfy the same boundary conditions.

Lastly we observe that $D_3 \leq 1$ for every time t: since it is assumed that $D_1 \leq 1$, $D_2 \leq 1$, equation IV.2. immediately renders:

$$D_3^2 \le (\sigma D_1 + (1-\sigma)D_2)^2 \le 1 \quad .$$

V. Physical Interpretation.

The techniques used to construct the control pair (φ_3, D_3) , in section IV, are elementary. The fact that this control pair exists, however, has considerable significance, inasmuch as the pair (φ, D) does not act as a linear control, but the associated control space is nevertheless convex.

Looking at the conventional problem, where $D \equiv 1$ for each D , we see that if W_1 and W_2 are in $\tilde{C}_0$ (or in $\tilde{C}_\tau$) then a trajectory W_3 between, and not equal to, W_1 and W_2 is in $C_0 - \tilde{C}_0$ (or in $C_\tau - \tilde{C}_\tau$); that is, it requires a diminution of thrust to obtain W_3.

Examples are illustrative. If W_1 and W_2 are in $\tilde{C}_0$, or $\tilde{C}_\tau$, and W_3 (for an arbitrary choice of σ in $(0,1)$ is obtained by the procedure given in section IV, and:

1. If $|\varphi_1-\varphi_2| \le 8^\circ$, then D_3 lies in the range $.997 < D_3 \le 1.000$, corresponding to baffling of less than .3% for all times, t.

2. If $|\varphi_1-\varphi_2| \le 25^\circ$, then D_3 lies in the range $.97 < D_3 \le 1.00$, corresponding to baffling of less than 3% for all times, t.

Thus small amounts of thrust baffling have a pronounced effect on the control space of the vehicle. It may eventually be worthwhile to study the possibility of actually adding such a control as a convenient way of increasing the maneuverability, mission flexibility, and the reliability of the space vehicle--especially if such a control can be made fuel-conservative.

In case these last remarks are not self-explanatory, we note that convexity in the control space results from the addition of many trajectories which are unobtainable at present. Also, this convexity should permit many space-vehicle guidance problems to be solved with linear algorithms, using precomputed trajectories, to obtain theoretically exact solutions. Since linear algorithms are usually more simple than others, they are economical of computer time and more accessible to small (sic: inboard) computers, in general. If, also, the control, D_1 is made fuel-conservative, then time optimality is no longer the sole criterion for fuel-preservation optimality; and it may be possible to choose a much "safer" trajectory than the time-optimal one, with little or no cost in added fuel consumption.

VI. Conclusion.

Various associated problems are outstanding. The physical meaning of the preceding results suggests that at each time, t, there is a one to one mapping of $\tilde{C}_0(t)$ (or $\tilde{C}_\tau(t)$, or $\tilde{C}_\tau(t) \cap \tilde{C}_0(t)$) onto a convex subset of E_4; but as of yet, no such mapping has been found. In practice, the control φ will have a bounded derivative, say $|\dot{\phi}| \leq B$. It is not clear whether the control space will remain convex when the bound, B, is imposed on $|\dot{\phi}|$, or if not, whether the procedure in section IV can be appropriately modified so that, for instance, $C_0(t)$ is convex at each time t (which does not imply that C_0 is convex). It has not yet been determined whether the results extend to the more general cases under investigation. The structure of the space $C_0 \cap (\cup \{C_\tau \mid \tau \in \mathfrak{T}\})$ has not been investigated.

The Chapel Hill research on this problem was terminated in its incipient stages. The results presented here were to have been the starting point for a (hopefully) thorough investigation of the problem. It is hoped that this paper will provide a useful background to the general control space problem, suggest a convenient and physically meaningful mathematical device with which to place the problem in a more tractable context, and aid in a physical interpretation of the results.

I. INTRODUCTION

In this paper, we outline a method for converting a problem in differential systems with non-linear controls to a problem in linear algebra. The method can be applied to obtain (possibly non-optimal) solutions of the two-point boundary condition problem, and alternatively to obtain a time optimal solution from a known, non-optimal solution. The flat earth problem is assumed.[1]

The method is inexact, and considerable research remains to be done in measuring and improving the accuracy of the approximations involved. Accuracy is limited only by the available computer facility; but it is still required to know how to obtain maximum efficiency with the available computer facility. Research to provide more efficient (or more useful) solutions to the associated problem in linear algebra is desirable also.

The method as outlined here, for two illustrative problems, incorporates two main procedures:

The first procedure is to develop a set of appropriately simple, linearly independent functions, γ_p , such that the desired control function, φ , can be accurately approximated by a linear combination of the γ_p's , multiplied by appropriate constants C_p :

$$\varphi \doteq \sum_{p=1}^{N} C_p \gamma_p \quad .$$

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1. J. J. [illegible] and [illegible], "Introduction into the Concept of the Adaptive [illegible]," George C. Marshall Space Flight Center Aeroballistics Internal Note, No. 21-60, (December 28, [illegible]).

The second procedure is to take a functional derivative, using variational methods; and then, by equating this derivative to a direction in which the trajectory associated with a somewhat arbitrary starting function is to be perturbed, we obtain a matrix formulation for determining the desired constant C_p's .

II. THE VARIATIONAL DERIVATIVE

Suppose that $\varphi(t)$ is a given function of the angle of thrust with respect to time. Let $W(\varphi,t) = \{x(\varphi,t),\ y(\varphi,t),\ u(\varphi,t),\ v(\varphi,t)\}$ be the associated trajectory, defined by the equations:

$$\dot{x}(\varphi,t) = u(\varphi,t)$$

$$\dot{y}(\varphi,t) = v(\varphi,t)$$

$$\dot{u}(\varphi,t) = T \sin \varphi(t)$$

$$\dot{v}(\varphi,t) = T \cos \varphi(t) - g$$

together with the initial value $W(\varphi,0)$ which W assumes when $t = 0$.

When φ is a known function, and $W(\varphi,0)$ is specified, the value which $W(\varphi,t)$ takes on when $t = \tau$ can be found by straightforward integration:

$$W(\varphi,\tau) = W(\varphi,0) + \int_0^\tau G(\varphi(s),s)\,ds \quad .$$

Suppose now that χ and Γ are both known functions, $W(\chi,\tau)$ has been computed as above; and it is desired to find the effect on W of incrementing χ by an amount $\epsilon\Gamma$, where ϵ is small.

Approximating, we have:

(1) $$\Delta[W(\chi,\tau),\epsilon\Gamma] = W(\chi+\epsilon\Gamma,\tau) - W(\chi,\tau) \doteq \epsilon\left(\frac{d}{d\sigma} W(\chi+\sigma\Gamma,\tau)\right)_{\sigma=0} .$$

with an error (here undetermined) dependent on the size of ϵ , for fixed Γ .

Now,

$$\Delta[W(\chi,\tau),\epsilon\Gamma] \doteq \epsilon\left(\frac{d}{d\sigma} W(\chi+\sigma\Gamma,\tau)\right)_{\sigma=0}$$

$$= \epsilon\frac{d}{d\sigma}\begin{pmatrix} x(\chi+\sigma\Gamma,\tau) \\ y(\chi+\sigma\Gamma,\tau) \\ u(\chi+\sigma\Gamma,\tau) \\ v(\chi+\sigma\Gamma,\tau) \end{pmatrix}_{\sigma=0}$$

$$= \epsilon\frac{d}{d\sigma}\begin{pmatrix} \int_0^\tau (\tau-s)\ T(s)\ \sin\ (\chi(s)+\sigma\Gamma(s))\ ds \\ \int_0^\tau (\tau-s)\ T(s)\ \cos\ (\chi(s)+\sigma\Gamma(s))\ ds \\ \int_0^\tau T(s)\ \sin\ (\chi(s)+\sigma\Gamma(s))\ ds \\ \int_0^\tau T(s)\ \cos\ (\chi(s)+\sigma\Gamma(s))\ ds \end{pmatrix}_{\sigma=0}$$

(where the terms not involving σ have been dropped)

$$
= \begin{pmatrix}
\int_0^\tau (e(\tau-s)\ T(s)\ \cos X(s))\ \Gamma(s)\ ds \\
- \int_0^\tau (e(\tau-s)\ T(s)\ \sin X(s))\ \Gamma(s)\ ds \\
\int_0^\tau (\epsilon T(s)\ \cos X(s))\ T(s)\ ds \\
- \int_0^\tau (\epsilon T(s)\ \sin X(s))\ T(s)\ ds
\end{pmatrix}
$$

(2)
$$= \dot{\Delta}[W(X,\tau),\epsilon\Gamma]$$

Equation II.2 defines an approximation. It is worth noting here that

(3)
$$\dot{\Delta}[W(X,\tau),\epsilon\Gamma] = \epsilon\dot{\Delta}[W(X,\tau),\Gamma] \quad ,$$

although the equality II.3 does not hold, in general, if the Δ's are not dotted.

If $\dot{\Delta}[W(X,\tau),\Gamma]$ is normalized, it corresponds to a sort of directional derivative; and if $|\epsilon\Gamma|$ is small,

(4)
$$\Delta[W(X,\tau),\epsilon\Gamma] \doteq \epsilon\dot{\Delta}[W(X,\tau),\Gamma]$$

is a good approximation.

III. AN APPROXIMATING FUNCTION

We will develop a typical approximating function. Our choice can be improved, but it is illustrative.

We are concerned with an interval, $[0,\tau]$. Let N be a positive integer. We define a step size, $h = \tau/N$, and for each integer k ,

t_k is the number kh. For each integer k, we define

$$\gamma_k(t) = t - t_{k-1} \quad \text{if } t \text{ is in } [t_{k-1}, t_k]$$
$$= t_{k+1} - t \quad \text{if } t \text{ is in } [t_k, t_{k+1}]$$
$$= 0 \quad \text{otherwise} .$$

For every sequence, C, of numbers, where $C = \{c_p\}_{p=0}^{N}$, we define $\Gamma(C,t)$ to be the function whose value at t is:

$$\sum_{p=0}^{N} c_p \gamma_p(t) \quad .$$

Then $\Gamma(C,t)$ is a polygonal function such that $\Gamma(C,t_p) = c_p$ if $0 \le p \le N$, and $\Gamma(C,t)$ can be expressed as the integral of a step function, all of whose jumps are at the points t_p, $0 \le p \le N$.

IV. A LINEAR ALGEBRAIC EXPRESSION FOR $\dot{A}$

Suppose that in II.1 and II.2, we take Γ to be a polygonal function of the form $\Gamma(C,t)$, with $C = \{c_p\}_{p=0}^{t}$. Then from II.4:

$$\Delta[W(\chi,\tau),\epsilon\Gamma] \doteq \epsilon\dot{\Delta}[W(\chi,\tau),\Gamma]$$

$$= \epsilon \begin{pmatrix} \int_0^\tau \left[(\tau-s)\, T(s) \cos \chi(s)\right] \left\{\sum_{p=0}^{N} C_p \gamma_p(s)\right\} ds \\ -\int_0^\tau \left[(\tau-s)\, T(s) \sin \chi(s)\right] \left\{\sum_{p=0}^{N} C_p \gamma_p(s)\right\} ds \\ \int_0^\tau \left[T(s) \cos \chi(s)\right] \left\{\sum_{p=0}^{N} C_p \gamma_p(s)\right\} ds \\ -\int_0^\tau \left[T(s) \sin \chi(s)\right] \left\{\sum_{p=0}^{N} C_p \gamma_p(s)\right\} ds \end{pmatrix}$$

$$= \epsilon \sum_{p=0}^{N} C_p \begin{pmatrix} \int_0^\tau (\tau-s)\, T(s)\gamma_p(s) \cos \chi(s)\, ds \\ -\int_0^\tau (\tau-s)\, T(s)\gamma_p(s) \sin \chi(s)\, ds \\ -\int_0^\tau T(s)\gamma_p(s) \cos \chi(s)\, ds \\ -\int_0^\tau T(s)\gamma_p(s) \sin \chi(s)\, ds \end{pmatrix}$$

Now for each control function, χ, defined on $[0,\tau]$, we define an associated $4 \times [N+1]$ matrix $M(\chi)$, whose components are as follows:

If $0 \leq k \leq N$, then

$$(1) \qquad M(X)_{1,k} = \int_0^\tau (\tau-s)\ T(s)\gamma_k(s)\cos X(s)\ ds$$

$$M(X)_{2,k} = -\int_0^\tau (\tau-s)\ T(s)\gamma_k(s)\sin X(s)\ ds$$

$$M(X)_{3,k} = \int_0^\tau T(s)\gamma_k(s)\cos X(s)\ ds$$

$$M(X)_{4,k} = -\int_0^\tau T(s)\gamma_k(s)\sin X(s)\ ds$$

It is noteworthy that each of the $4N$ integrals involved in this definition is simplified by the fact that $\gamma_k(t) = 0$ if t is not in $[t_{k-1}, t_{k+1}]$. The computing task involved is approximately that of integrating eight times across the interval $[0,\tau]$, irrespective of the size of N.

In practice, an approximation will occur when the elements of the matrix $M(X)$ are computed. Disregarding this latter approximation, we have:

$$(2') \qquad \Delta[W(X,\tau),\epsilon\Gamma] \doteq \epsilon\dot{\Delta}[W(X,\tau),\Gamma] = \epsilon M(X)C \ .$$

where $\Gamma = \Gamma(C,t)$ and $C = [C_p]_{p=0}^{N}$ is regarded as a column vector.

This equation, IV.2', may be rewritten now to emphasize its important feature:

$$(2) \qquad \Delta[W(X,\tau),\epsilon\Gamma(C,t)] \doteq \epsilon M(X)C$$

V. THE TWO POINT BOUNDARY CONDITION

This is the first of two examples in which the approximation IV.2 is applied to obtain a linear algebraic formulation of a problem in differential equations involving non-linear controls. We take a familiar problem:

The space vehicle is required to have vehicle coordinates X_0 at a specified time, $t = \tau_0$. X is the function of time with the property that if the vehicle has vehicle coordinates $X(t)$ at time t, then it will reach X_0 at τ_0, by unpowered flight. $X(t)$ is specified to be a single point, for each time t; in practice, $X(t)$ may be chosen in the center of a set of permissible points.

We assume that it is known that there is some control function ϕ such that the associated trajectory $W(\phi,t)$ has cutoff time τ and $W(\phi,\tau) = X(\tau)$. That is, we assume that the question of existence has been decided, and it is desired to compute one of the solutions that are known to exist.

We choose a starting function, φ, subject to mild restrictions: If φ is slowly varying, $M(\varphi)$ (cf. IV.1) can be computed more accurately; if φ is fairly near the final solution obtained, the method will work more efficiently. The vehicle coordinates at $t = 0$ are known, hence we can compute them at $t = \tau$.

We now wish to find a function $\Gamma(C,t)$, and if necessary a number, ϵ, $0 < \epsilon \leq 1$, such that $W(\varphi+\epsilon\Gamma(C,t),\tau)$ is closer to $X(\tau)$ than $W(\varphi,\tau)$. $\Gamma(C,t)$ is determined by our choice of C; and to obtain C we solve the equation:

$$M(\varphi)C = X(\tau) - W(\varphi,\tau) \tag{1}$$

We next choose ϵ, $0 < \epsilon \leq 1$, so that

$$\Delta[W(\varphi,\tau),\epsilon\Gamma(C,t)] \doteq \epsilon M(\varphi)C , \tag{2}$$

which is the same as IV.2, is a good approximation (we do not attempt, in this paper, to exposit criteria for evaluating whether an approximation of this type is "good").

When C and ϵ have been so determined, we have:

$$\Delta[W(\varphi,\tau),\epsilon\Gamma(C,t)] \doteq \epsilon(X(\tau) - W(\varphi,\tau)) .$$

which yields (note def. II.1):

$$W(\varphi+\epsilon\Gamma(C,t),\tau) \doteq W(\varphi,\tau) + \epsilon(X(\tau)-W(\varphi,\tau)) \quad . \tag{3}$$

If $\epsilon = 1$, and the approximation in step 3 is sufficiently good, the problem is solved, since in that case, V.3 reduces to

$$W(\varphi+\epsilon\Gamma(C,t),\tau) \doteq X(\tau) \quad ,$$

where the approximation is supposed good.

If $\epsilon = 1$, but the approximation V.3 is not sufficiently accurate, or if $\epsilon < 1$, the procedure may be repeated, using $\varphi + \epsilon\Gamma$ as the new starting function.

VI. THE TIME OPTIMAL PROBLEM

Suppose that we have obtained (by the calculations outlined in section V, or otherwise) a control function, φ , whose associated trajectory $W(\varphi,t)$ has cutoff at $X(\tau_0)$ when $t = \tau_0$, which is not the optimal cutoff time, with $X(t)$ as in section V.

That is, $W(\varphi,t)$ is a trajectory which completes the mission successfully, but not time-optimally with respect to cutoff.

We wish to find a function Γ and an appropriate factor, ϵ , so that at some time τ , antecedent to τ_0 , we will have $W(\varphi+\epsilon\Gamma,\tau) \doteq X(\tau)$, the approximation within the accuracy requirements of the problem.

Proceeding, we have, for τ near τ_0 :

$$W(\varphi+(\tau-\tau_0)\Gamma,\tau) - W(\varphi,\tau_0) = W(\varphi+(\tau-\tau_0)\Gamma,\tau) - W(\varphi+(\tau-\tau_0)\Gamma,\tau_0)$$

(1)

$$+ W(\varphi+(\tau-\tau_0)\Gamma,\tau_0) - W(\varphi,\tau_0) \quad .$$

If $|\tau-\tau_0|$ is small,

(2) $$W(\varphi+(\tau-\tau_0)\Gamma,\tau) - W(\varphi+(\tau-\tau_0)\Gamma,\tau_0) \doteq (\tau-\tau_0)\dot{W}(\varphi+(\tau-\tau_0)\Gamma,\tau_0)$$

is a good approximation (we still have not developed specific criteria to determine what is a good approximation). If we further provide that Γ is zero in a neighborhood of τ_0 , (i.e., by setting $C_N = C_{N-1} = 0$, where $\Gamma(C,t)$), then

(3) $$\dot{W}(\varphi+(\tau-\tau_0)\Gamma,\tau_0) = \dot{W}(\varphi,\tau_0) \quad .$$

Also, if $|\tau-\tau_0|$ is small, from IV.2 .

(4) $$W(\varphi+(\tau-\tau_0)\Gamma,\tau_0) - W(\varphi,\tau_0) \doteq (\tau-\tau_0)M(\varphi)C$$

is a good approximation, where $\Gamma = \Gamma(C,t)$. Now combining VI.2,3,4, we have

(5) $$W(\varphi+(\tau-\tau_0)\Gamma(C,t),\tau) - W(\varphi,\tau_0) \doteq (\tau-\tau_0)\dot{W}(\varphi,\tau_0) + (\tau-\tau_0)M(\varphi)C \quad ,$$

with the approximation good if $|\tau-\tau_0|$ is small. Note that we have make no requirements on C so far, except that $C_N = C_{N-1} = 0$.

Now we wish to equate the left-hand side of VI.5 with $X(\tau) - X(\tau_0)$. Since we have not determined τ , except that it lies near τ_0 , we make the final approximation:

(6) $$X(\tau) - X(\tau_0) \doteq (\tau-\tau_0)\dot{X}(\tau_0) \quad ,$$

in order to obtain $X(\tau) - X(\tau_0)$ as a function linear in $(\tau-\tau_0)$.

Now we can approximately equate the left-hand sides of VI.5 and VI.6 by setting the right-hand sides of these equations equal. We then transpose to obtain:

(7) $$(\tau-\tau_0)M(\tau)C = (\tau-\tau_0)\left[\dot{X}(\tau_0)-\dot{W}(\varphi,\tau_0)\right] .$$

$(\tau-\tau_0)$ may be cancelled from both sides of equation VI.7, and the resulting matrix equation determines C , regardless of the choice of τ ; i.e.,

$$M(\tau)C = \left[\dot{X}(\tau_0)-\dot{W}(\varphi,\tau_0)\right] .$$

Consequently, we could have computed C at the outset. Our choice of τ is governed only by a desire to keep the approximations VI,2,4,6 within our accuracy requirements.

Having chosen τ , we can improve our choice of C by reverting to the calculations of section V, if desirable. Otherwise, we can repeat the procedure of this section until we are suitably near the optimal cut-off time.

VII. RESEARCH PROPOSALS

The method outlined in the previous sections is quite general; and it can be adapted to other problems in control theory than the two by which we have illustrated it. It would naturally be desirable to express the method in a more general setting, to facilitate its application to problems which may arise in the future.

Of more immediate interest is the problem of developing specific steps which will enable us to use the method, as outlined, in the form of a sequence of numerical algorithms. The utility of the method as a computational device will be suppressed until we have achieved a convenient means of measuring and restricting the error associated with our various approximations. At the same time, the method should be accessible to the available computing facility.

In theory, we can reduce our error as much as we wish, if we are reckless of the cost in computer time necessary to use the method. In practice, our computer time is limited; and we will have to choose a balance between speed and accuracy.

The problem is difficult, and it will involve some complex numerical studies. To each variation in step size, h , (as in section III) and choice of ϵ in approximations of the form II.4, or in related approximations such as VI.7, we must assign some value for computing speed and another value for accuracy, until we have covered whatever proves to be the range of interest.

The polygonal path approximation we have used suffers the disadvantage that it cannot be used as a guidance function (the derivative need not exist at the t_k) ; and some rounding will be necessary.

We should determine when it is better to take two small steps with small error than two large steps across the same interval, letting the second step improve upon the accuracy of the first. That is, we should determine when, if ever, it is desirable to take $\epsilon < 1$ in approximations of the form II.4.

It appears that equations of the form IV.2 can always be solved by linear programming techniques; but it is not at all clear that linear programming will provide the most desirable answer, if more than one answer is available. Other approaches to the linear algebraic problem should be considered, with the point of view of finding "safe" solutions (trajectories on which small deviations will not drastically increase cut-off time, or ruin the mission).

A thorough search of the relevant literature has not been made.

Another problem is that of extending this method to more general settings: in particular, to the round-earth problem. This extension has not, as yet, been attempted. The power of the method depends in large measure upon being able to express in closed form the trajectory associated with a specified guidance function. However, the solution to the round-earth problem can be found as a perturbation of the solution of the flat-earth problem, and it may be possible to incorporate the appropriate perturbation procedures into the method already given, to produce a single computational algorithm for the whole problem. Such an investigation would be a logical continuation of the research already proposed.

The method outlined appears likely to be of great usefulness and relevance, both to the problems by which it is illustrated, and to problems which may arise in the field of non-linear differential controls.

PRELIMINARY INVESTIGATIONS OF A METHOD OF DETERMINING BEST PREDICTION EQUATIONS

1. Introduction.

A frequent problem encountered in computer work is that of finding the best equation to be used for predicting the value of a dependent variable, y, knowing the values of several independent variables $z_1, z_2, \cdots, z_r$. This equation usually takes the form

$$y = b_1X_1(\underline{z}) + b_2X_2(\underline{z}) + b_mX_m(\underline{z}) + e \tag{1}$$

where the $X_j(\underline{z})$'s are known functions of the independent vector $\underline{z}$, the b_j's are parameters to be determined from observed data, and e is the error of prediction. While it is generally considered adequate to estimate the b's by a least squares method, it is often also desirable to include in the equation only as many variables as are needed to predict y adequately. Thus arises the question of which X's may be excluded without adversely affecting the utility of the prediction equation.

2. Present Stepwise Methods.

The question posed above is by no means a new problem. Several stepwise methods for deciding which variables to include in a prediction equation have been suggested and discussed in the literature. Probably the most widely used method is that suggested by Efroymson, called by some the "step forward and review" method. This method introduces a new variable (now referring to the X's as independent variables instead of the z's) into the equation at each step, choosing the new variable so that the additional reduction in the error sum of squares is a maximum. If there are p variables in the equation, the decision to include a $(p+1)^{th}$ variable is made if the additional reduction in the error sum of squares is significant according to the usual statistical F test. After the addition of a new variable, the contributions of the other variables are re-examined to determine whether one of the original p variables can now be excluded from the equation without causing a significant change in the error sum of squares. The procedure halts when no variable can be dropped and no new variable can be added to the equation. All of the other stepwise methods are variations of this same theme--at each step a new variable may be added or one of the old variables deleted according to some criterion of significance. Thus these stepwise methods arrive at similar points: we have a p-variable equation and alteration of this equation by deleting one of the p variables or by adding one new variable does not produce a significant improvement.

3. The Proposed Method.

At the conclusion of any of the stepwise procedures mentioned above there are several important questions left unanswered. Among these are:

(a) Since we seem to be stuck with a p-variable equation, is there any other combination of p variables that yield a better prediction (as measured by the error sum of squares) than the p-variable solution to the stepwise procedure?

(b) Is there any combination of (p+1) variables (not necessarily derivable by adding one new variable to the p variables already in the equation that yields a significantly better predicition than the stepwise solution?

(c) Can the prediction be significantly improved by adding more than one variable to those in the stepwise solution?

The proposed method answers the first two questions by operating on the principle that if we must have a p-variable equation, then we should use that combination of p variables that minimizes the error sum of squares. Thus at each stage we should be comparing the "best" combination of (p+1) variables with the "best" combination of p variables where "best" is defined in terms of minimum error sum of squares. The answer to the third question will be determined by the criterion for stopping the stepwise procedure.

The key to the whole procedure is the development of a feasible algorithm to find that combination of p variables out of a possible m variables that minimizes the error sum of squares. This could, of course, be accomplished by evaluating every possible combination of

p variables, but that is out of the question of feasibility since the number of combinations increases excessively with m. The suggested algorithm will be developed by demonstrating the equivalence of the least squares technique and the linear programming problem of maximizing the function

$$L = g_1 b_1 + g_2 b_2 + \cdots + g_m b_m$$

subject to the restrictions

$$A \underline{b} \leq \underline{g}$$

where the vector $\underline{b}$ is the solution vector, $\underline{g}$ is the right-hand side of the normal equations of the least squares technique and the matrix A results from a modification of the normal equations. The restriction that there be exactly p variables in the equation can be imposed by adding new restrictions or by altering the simplex algorithm for solving the linear programming problem.

Continuation of the investigation of this method will include formalizing the procedure and proving its results, then demonstrating its utility and feasibility by empirical comparison with present stepwise procedures and with the algorithm for generating all possible combinations.

MATHEMATICAL SYMBOL MANIPULATION BY DIGITAL COMPUTER

At the expiration of the Contract NAS8-1528 on 30 September 1964, The Computation Center was completing work on a series of general algorithms that would simplify any given algebraic expression through the collection or cancellation of like terms and by factoring out common terms. These algorithms were based on the repeated application of the communicative laws of addition and multiplication and of the distributive law and were general in the sense that they were not restricted to the manipulation of a restricted class of expressions, e. g., polynomial expressions, etc. In addition, they were designed to be used in conjunction with the previously completed algorithm for performing analytic differentiation in order that a general symbol manipulation package could be developed. Work on these algorithms was completed in October, 1964 and development of a computer program for the implementation of these algorithms was begun. These programs were basically completed early in 1965.

In addition to the above research the Computation Center had been assigned to work on a specific symbol manipulation problem that had been encountered in the work of another contractor. This problem can best be explained in terms of the following three expressions:

Expression 1.

$$[illegible] = [illegible] \{ [illegible] \left[2 U_3 V_3 \, [illegible] \, 2\varphi - (U_3^2 - V_3^2) \sin 2\varphi \right]$$

$$+ 3 \left[1 - \frac{3}{2} (U_3^2 + V_3^2) - \frac{3}{2} (U_3^2 - V_3^2) \cos 2\varphi - 3 U_3 V_3 \, [illegible] \, 2\varphi \right] \}$$

Expression 2

$$[illegible] = 2 \left[\sin \varphi \, [illegible] (1 + p_0) (2 + p_0) - q_0 \, [illegible] \right] - 9 q_0 \, [illegible] \, \varphi$$

$$[illegible] = 2 (1 + p_0) [illegible] \varphi + [illegible] - [illegible] (1 + p_0)^2 + [illegible] (1 + p_0) q_0 \sin \varphi$$

$$p = p_0 \cos \varphi - q_0 \sin \varphi$$

$$q = p_0 \sin \varphi + q_0 \, [illegible] \, \varphi$$

$$p^2 + q^2 = [illegible] = p_0^2 + q_0^2$$

Expression 3

$$\left[\frac{3}{2} (U_3^2 + V_3^2) - 1 \right] [illegible]$$

$$- [illegible] \varphi \left\{ U_3 V_3 \left(2 q_0 - \frac{3}{4} p_0 q_0 - \frac{7}{8} p_0 [illegible] - \frac{3}{4} q_0 \right) - (U_3^2 - V_3^2) \left(\frac{1}{2} + \frac{5}{4} p_0 + \frac{7}{8} p_0^2 \right. \right.$$

$$\left. \left. + \frac{1}{8} p_0^3 + \frac{9}{8} q_0^2 \right) + \left[\frac{3}{2} (U_3^2 + V_3^2) - 1 \right] \left(-2 - p_0 + \frac{1}{2} p_0^2 - \frac{1}{2} p_0^3 - \frac{1}{2} q_0^2 - p_0 q_0^2 \right) \right\}$$

$$+ \cos 2\varphi \left\{ - U_3 V_3 \left(\frac{3}{2} q_0 + \frac{5}{2} p_0 q_0 \right) - (U_3^2 - V_3^2) \left(\frac{3}{2} + \frac{7}{4} p_0 + \frac{1}{4} p_0^2 - \frac{5}{8} q_0^2 \right) \right.$$

$$\left. - \left[\frac{3}{2} (U_3^2 + V_3^2) - 1 \right] \left(p_0 + p_0^2 + \frac{1}{2} q_0^2 \right) \right\}$$

$$+ \cos [illegible] \varphi \left\{ U_3 V_3 \left(2 q_0 - \frac{29}{7} p_0 q_0 - \frac{1}{2} p_0^2 q_0 + \frac{11}{8} q_0^3 \right) \right.$$

$$+ (U_3^2 - V_3^2) \left(\frac{7}{2} + \frac{11}{4} p_0 - \frac{1}{7} p_0^2 + \frac{17}{2} q_0^2 + \frac{5}{4} p_0^3 - p_0 q_0^2 \right)$$

$$\left. + \left[\frac{3}{2} (U_3^2 - V_3^2) - 1 \right] \left(\frac{1}{2} q_0^2 - \frac{1}{2} p_0^2 - \frac{1}{2} p_0^3 + p_0 q_0^2 \right) \right\}$$

$$+ \cos [illegible] \varphi \left\{ U_3 V_3 \left(\tfrac{1}{2} q_0 + \tfrac{3}{2} p_0 q_0 \right) + \tfrac{5}{8} \left(U_3^2 - V_3^2 \right) \left(2 p_0 + 2 p_0^2 - q_0^2 \right) \right\}$$

$$+ \cos [illegible] \varphi \left\{ \frac{U_3 V_3}{9} \left(2 [illegible] p_0 q_0 + 2 [illegible] p_0^2 q_0 - 5 q_0^3 \right) + \left(U_3^2 - V_3^2 \right) \left(\tfrac{1}{4} p_0 [illegible] + \tfrac{1}{2} p_0^2 - \tfrac{5}{4} p_0 q_0^2 - \tfrac{5}{7} q_0^4 \right) \right\}$$

$$+ \sin \varphi \left\{ - U_3 V_3 \left(1 + \tfrac{5}{2} p_0 + \tfrac{3}{2} p_0^2 - \tfrac{5}{2} q_0^2 - \tfrac{3}{2} p_0 q_0^2 \right) + \left(U_3^2 - V_3^2 \right) \left(- q_0 + \tfrac{1}{2} p_0 q_0 \right.\right.$$

$$\left.\left. + \tfrac{1}{2} p_0^2 q_0 + \tfrac{1}{2} q_0^3 \right) + \left[\tfrac{3}{2} \left(U_3^2 + V_3^2 \right) - 1 \right] \left(- A q_0 + \tfrac{1}{4} p_0^2 q_0 + \tfrac{3}{2} q_0^3 \right) \right\}$$

$$+ \sin 2\varphi \left\{ 3 U_3 V_3 \left(-1 + q_0^2 + p_0^2 - \tfrac{1}{4} p_0 q_0 \right) + 5 \left(U_3^2 - V_3^2 \right) \left(q_0 + \tfrac{1}{2} p_0 q_0 \right) \right.$$

$$\left. + \left[\tfrac{3}{2} \left(U_3^2 + V_3^2 \right) - 1 \right] \left(2 q_0 + 3 p_0 q_0 \right) \right\}$$

$$+ \sin 3\varphi \left\{ U_3 V_3 \left(7 + \tfrac{11}{2} p_0 + \tfrac{7}{4} q_0^2 - \tfrac{1}{4} p_0^2 + \tfrac{2}{4} p_0^2 + \tfrac{3}{2} p_0 q_0^2 \right) \right.$$

$$\left. + \left(U_3^2 - V_3^2 \right) \left(- q_0 - \tfrac{11}{11} q_0^3 + \tfrac{7}{16} p_0 q_0 + \tfrac{3}{4} p_0 q_0 \right) + \left[\tfrac{3}{2} \left(U_3^2 + V_3^2 \right) - 1 \right] \left([illegible] \right) \right\}$$

$$+ \sin 4\varphi \left\{ U_3 V_3 \left(6 p_0 + [illegible] p_0^2 - 3 q_0^2 + \tfrac{3}{4} p_0 q_0 \right) - \left(U_3^2 - V_3^2 \right) \left(3 q_0 + \tfrac{1}{2} p_0 q_0 \right) \right\}$$

$$+ \sin 5\varphi \left\{ 5 U_3 V_3 \left(\tfrac{1}{2} p_0 [illegible] + \tfrac{1}{4} p_0^2 - \tfrac{1}{4} q_0^2 - \tfrac{1}{2} p_0 q_0^2 \right) + 5 \left(U_3^2 - V_3^2 \right) \left(- \tfrac{1}{4} p_0 q_0 - \tfrac{5}{16} p_0 q_0 + \tfrac{1}{4} q_0^3 \right) \right\}$$

The problem was to make the variable substitutions given in expression 2 into expression 1 and then perform the necessary multiplications, collection and cancellation of like terms, recognition of trigonometric identities, and factoring to arrive at expression 3. It was necessary to perform these operations on large sets of such expressions with absolute accuracy.

In order to do this work on a computer the following algorithms, applied in the order listed, would be needed.

1. An algorithm to substitute the equations in expression 2 for the defined variables appearing in expression 1.

2. An algorithm that would multiply out all parenthesized terms.

3. Algorithms that would simplify the expression resulting from the above multiplication, by the collection or cancellation of like terms.

4. An algorithm that would simplify the products of trigonometric terms by the application of the appropriate trigonometric identifies.

5. An algorithm to factor out like expressions.

Since the algorithm needed for 1. is easily developed and not immediately necessary, work on it was postponed. The algorithms needed for steps 3. and 5. are the same ones that were developed in the previously described symbol manipulation work.

Upon completing the simplification and factoring algorithms in October, the development of a multiplication algorithm was begun. Work on this algorithm was completed in March and as this report period ended, the necessary computer program for the implementation of the algorithm was nearing completion.

As the work on multiplication moved into the implementation stage, development work was begun on the algorithms required to perform the trigonometric simplifications. By the end of the report period, this work was well underway.

A FEASIBILITY STUDY ON THE DEVELOPMENT OF COMPUTER ALGORITHMS TO PERFORM SYMBOLIC INTEGRATION

I. Introduction.

To date the Computation Center has developed and implemented an algorithm for performing analytic differentiation by computer and algorithms to do algebraic simplification of equations through the collection and cancellation of common terms. Work is currently underway on the problems of multiplying out complex parenthesized expressions, variable transformations, and simplifications of expressions by limited trigonometric identities. It is planned to extend the scope of the current projects and efforts to include symbolic integration, simplification of expressions by more general trigonometric identities, manipulation of series, and the incorporation of these various algorithms into the body of an algebraic compiler so as to add analytical capabilities to the now existing computational capabilities of these compilers.

The remainder of this report discusses the proposed approaches for attacking the problem of symbolic integration on a computer.

II. Operational Antiderivative Techniques.

One method of approach to this problem that offers partial results is through the use of the operational technique for the antiderivative operator, D^{-1}. These methods are particularly useful with a fairly restricted set of functions for which integration by parts is normally required.

Suppose F is a function which can be expressed as a sum of products:

$$F = \sum_{p=1}^{N} G_p H_p ,$$

where G_p is a polynomial for each p, of degree M_p, and where H_p is a function whose first $M_p + 1$ integrals are known.

Then the integral (antiderivative), $D^{-1}F$, of F can be expressed as:

(1)
$$D^{-1}F = D^{-1}\sum_{p=1}^{N} G_p H_p = \sum_{k=0}^{M_p} (-1)^k D^{k} G_p \, D^{-k-1} H_p = \sum_{p=1}^{N}\sum_{k=0}^{M_p} (-1)^k D^{k} G_p \, D^{-k-1} H_p$$

This is a finite terminating integrations-by-parts formula expressed in operator form. Successive integrations by parts result in a terminating, hence exact, solution since each G_p is a polynomial. Exactness results only when G_p, for each p, is a polynomial. G_p is a polynomial if and only if there is some integer M_p such that if $k > M_p$, $D^k G_p = 0$.

Functions of the form indicated above are the only ones found to date for which the use of operator formulae has been unambiguously useful.

EXAMPLE:

Integrate $x^2 e^x$:

$$\int x^2 e^x dx = D^{-1} e^x x^2 = e^x (D+1)^{-1} x^2$$

$$(D+1)^{-1} = 1 - D + D^2 - D^3 + \cdots$$

$$e^x (D+1)^{-1} x^2 = e^x (1 - D + D^2 - D^3 \cdots) x^2 = e^x (x^2 - 2x + 2)$$

Hence

$$x^{2} e^{x} dx = e^{x} x^{2} (x+2) + \text{constant of integration} .$$

III. An Extension of the Operational Technique.

Formula I may be applied to obtain inexact solutions of

$$X = D^{-1}F \quad ,$$

when F is expressible as a sum,

$$F = \sum_{p=1}^{k} \bar{G}_p H_p \quad ,$$

and there is a positive number $\delta < 1$ and a positive number B such that for sufficiently large N,

2 $$|D^{N} \bar{G}_p| \leq B\delta^{N}$$

(on the interval under consideration).

Formula I is used to obtain an inexact solution once a suitable polynomial approximation G_p, have been obtained for $\bar{G}_p$, for each p.

Such procedures can possibly be standardized, but there are various drawbacks: Applicable computer algorithms must be designed to successfully test for the inequality (2), to formulate the approximating polynomials, the then to apply Formula I. However, certain classes of functions can be systematically included in the class of functions satisfying the inequality (2) (e.g., sin kx, cos kx, sinh kx, cosh kx for k < ... and log kx for x > 1, and e^{kx} for k < 1), and hence this extension could prove fruitful.

IV. A Case in Which the Operational Method Fails.

Suppose F is a function which can be expressed as a product of the form $F = e^{\varphi}G$, where G is integrable and φ is differentiable. Then

$$D^{-1}F = D^{-1}e^{\varphi}G = e^{\varphi}[\varphi' + D]^{-1}G$$

Now $[\varphi' + D]^{-1}G$ is the function H such that $(\varphi' + D)H = G$. H may be solved formally: $H = e^{-\varphi}\int e^{\varphi}G$, which yields

$$D^{-1}F = e^{\varphi}[\varphi' + D]^{-1}G = e^{\varphi}e^{-\varphi}\int e^{\varphi}G = \int e^{\varphi}G = D^{-1}F \; .$$

The formal procedure outlined does not give a simple form for $D^{-1}F$, unless H, in equation (3), has a simple form. This is the case when G is a polynomial, and in this case $D^{-1}F$ can be evaluated by the use of equation (1). Otherwise

$$H = e^{-\varphi}(D^{-1}F)$$

and the problem of evaluating $D^{-1}F$ still remains.

V. Other Approaches to the Problem.

It may be possible to abstract or extrapolate, from tables of integrals, a fairly small collection of algorithms which will cover a relatively large class of integrals. If such rules can be found, they do not currently appear to be inherent in the formalism of the operational calculus. Such a study may be fruitful, however, independent of the role played by operational methods.

The operational calculus appears to be a superior method for computing exact integrals only in certain integration-by-parts applications. It gives promise of being a useful device for obtaining superior approximations, expressed in terms of known functions, to a fairly wide class of common functions.